The geometry of involutions of algebraic groups and of Kac-Moody groups

International Workshop on Algebraic Groups, Quantum Groups and Related Topics

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Overview

- Groups with a root datum
- Buildings
- Unitary forms
- Flip-flop systems and Phan geometries
- Properties and applications of flip-flop systems
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Chevalley groups: $\text{SL}_{n+1}$

Starting point: Chevalley groups. These are essentially determined by

1. a field $\mathbb{F}$ and
2. a (spherical) root system (more specifically, a root datum).

Root systems can be described and classified by Dynkin diagrams.

Example

$G = \text{SL}_{n+1}(\mathbb{F})$ corresponds to root system of type $A_n$ with this diagram:

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1 ---- 2 ------ n-1 ---- n
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(Also true for $\text{PSL}_{n+1}$; one needs a root datum to distinguish between them.)

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For algebraically closed fields one obtains connected semi-simple linear algebraic groups; for finite fields (untwisted) finite groups of Lie type.
Let \( n = 2 \) and \( G = \text{SL}_3(\mathbb{F}) \). The associated root system \( \Phi \) of type \( A_2 \):

\[
\begin{align*}
\beta & \quad \alpha + \beta \\
-\alpha & \quad \alpha \\
-\alpha - \beta & \quad -\beta
\end{align*}
\]

To each root \( \rho \in \Phi \) a root group \( U_\rho \cong (\mathbb{F}, +) \) of \( G \) is associated:

\[
U_\alpha = \langle \begin{pmatrix} 1 & * & 0 \\ 1 & 0 & 1 \end{pmatrix} \rangle, \quad U_\beta = \langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & * & 1 \end{pmatrix} \rangle, \quad U_{\alpha + \beta} = \langle \begin{pmatrix} 1 & 0 & * \\ 1 & 0 & 1 \end{pmatrix} \rangle, \quad U_{-\alpha} = (U_\alpha)^{-1}, ...
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The root groups, the (commutator) relations between them and the torus \( T := \bigcap_{\rho \in \Phi} N_G(U_\rho) \) (diagonal matrices in \( G \)) determine \( G \) completely.
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Kac-Moody groups generalize Chevalley groups in a natural way. Again take ...

1. a field $\mathbb{F}$ and
2. a root system (root datum) whose Dynkin diagram has edge labels in $\{3, 4, 6, 8, \infty\}$.

(Again: need root datum, not just root system, to distinguish $\text{SL}$ from $\text{PSL}$.)

Example

Let $\mathbb{F}[t, t^{-1}]$ denote the ring of Laurent polynomials over $\mathbb{F}$. $G = \text{SL}_{n+1}(\mathbb{F}[t, t^{-1}])$ is a Kac-Moody group over $\mathbb{F}$ with root system of type $\tilde{A}_n$:

![Dynkin diagram]

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![Dynkin diagram](image)

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![Dynkin diagram of $\widetilde{A}_n$]

**Remark:** In general, Kac-Moody groups are not linear.
To obtain the root system of type $\tilde{A}_n$ we add a new root corresponding to the lowest root in $A_n$. For $n = 2$, we get a new root $\gamma$ corresponding to $-\alpha - \beta$.

The positive fundamental root groups now are:

$$U_\alpha = \langle \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \end{pmatrix} \mid a \in \mathbb{F} \rangle,$$
$$U_\beta = \langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & a & 0 \end{pmatrix} \mid a \in \mathbb{F} \rangle,$$
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The negative root groups can be obtained from the positive ones by applying the Chevalley-Cartan involution of $G$: Transpose, invert and swap $t$ and $t^{-1}$, hence

$$U_{-\gamma} = \langle \begin{pmatrix} 1 & 0 & -at^{-1} \\ 1 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{F} \rangle$$

and $U_\alpha, U_\beta$ as before.

$G$ is generated by its root groups.

Important consequence: The groups $U_+ = \langle U_\rho \mid \rho \in \Phi_+ \rangle$ and $U_- = \langle U_\rho \mid \rho \in \Phi_- \rangle$ are no longer conjugate to each other.
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Root groups in Kac-Moody groups

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What is a building

Let $G$ be a group with root datum.

The building $C(G)$ of $G$ can be realized as . . .

- ...a homogeneous space $G/B$, where $B = N_G(U)$ and $U$ is generated by all positive root groups.

  Example: For $G = \text{SL}_{n+1}(\mathbb{F})$,
  - $U$ is the group of unit upper triangular matrices and
  - $B$ is the group of upper triangular matrices.

- . . .CAT(0)-spaces, an incidence geometry, a Chamber system, . . .

- . . .a simplicial complex: Take as simplices all proper subgroups of $G$ containing $B$, ordered by reverse inclusion.

Careful: One group may act on several buildings. But the choice of a system of root groups resp. the group $B$ determines the building.
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Careful: One group may act on several buildings. But the choice of a system of root groups resp. the group $B$ determines the building.
Leg $G$ be a group with root datum, denote by $C = C(G)$ its associated building and by $(W, S)$ its Coxeter system.

Some properties of $C$:

- Labeled simplicial complex, with labels from $S \rightarrow$ every simplex has a type.
- System $A$ of subcomplexes called apartments, each isomorphic to the Coxeter complex of $(W, S)$. Any two simplices are contained in at least one apartment.
- Weyl-distance $\delta : C \times C \rightarrow W$ assigns “distances” to pairs of simplices.
- Numerical distance $l : C \times C \rightarrow \mathbb{N}$ defined by $l(\sigma_1, \sigma_2) := l(\delta(\sigma_1, \sigma_2))$.
- Building is called spherical if $l$ is bounded $\rightarrow$ notion of opposite simplices.
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Let $G$ be Chevalley / Kac-Moody group over $\mathbb{F}$, and $\sigma \in \text{Aut}(\mathbb{F})$ with $\sigma^2 = \text{id}$. Let $\theta$ be the composition of the Chevalley-Cartan involution of $G$ with $\sigma$. For $\text{SL}_n(\mathbb{F})$:

$$\theta : x \mapsto (\sigma(x)^T)^{-1}.$$ 

Then $K := \text{Fix}_G(\theta)$ is called ($\sigma$-)unitary form of $G$.

Examples

<table>
<thead>
<tr>
<th>$G$</th>
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<tbody>
<tr>
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- Let $G$ be Chevalley / Kac-Moody group over $\mathbb{F}$, and $\sigma \in \text{Aut}(\mathbb{F})$ with $\sigma^2 = \text{id}$. Let $\theta$ be the composition of the Chevalley-Cartan involution of $G$ with $\sigma$. For $\text{SL}_n(\mathbb{F})$:
  $$\theta : x \mapsto (\sigma(x)^T)^{-1}.$$  
- Then $K := \text{Fix}_G(\theta)$ is called $(\sigma)$-unitary form of $G$.

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  \theta : G/B_+ \to G/B_+ : xB_+ \mapsto \theta(xB_+)g = \theta(x)B_-g = \theta(x)gB_+.
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Applications

- Phan type theorems (Bennett, Devillers, Gramlich, Hoffman, H., Mühlherr, Nickel, Shpectorov)
- New lattices in Kac-Moody groups (Gramlich, Mühlherr)
- Automorphisms of unitary forms of Kac-Moody groups (Kac, Peterson; Caprace; Gramlich, Mars)
- Representation theory (Devillers, Gramlich, Mühlherr, Witzel):
  Generalize Solomon-Tits theorem
- Generalized Iwasawa decomposition (De Medts, Gramlich, H.):
  $G$ split conn. reductive $F$-group / Kac-Moody group over $F$. When does $G_F$ admit a decomposition $G_F = K_F B_F$ (where $K$ is centralizer of an involution)?
  (Inspired by Helminck & Wang, 1993.)
- Finiteness properties (Caprace, Devillers, Gramlich, H., Mühlherr, Witzel)
Let $\theta$ be an involutory almost-isometry of a building $C$.

For $\sigma \in C$ the local flip-flop system $C^\theta_\sigma$ consists of simplices in $\text{lk} \, \sigma$ for which the numerical $\theta$-distance is maximal among all simplices in the link.

Call $(C, \theta)$ a good pair if for all corank-2 simplices $\sigma \in C$, $C^\theta_\sigma$ is path connected and “allows direct ascent”.

**Theorem (Gramlich, H., Mülhlherr 2008)**

If $(C, \theta)$ is a good pair, then $C^\theta$ is path connected and pure, i.e., all its maximal simplices have equal type $J \subset S$. Moreover $C^\theta$ is residually connected, hence there exists an associated incidence geometry, the Phan geometry.

**Example (Bennet, Shpectorov)**

Let $\theta$ be a twisted Chevalley involution of $\text{SL}_n(\mathbb{F})$, $n \geq 3$ and $(n, \mathbb{F}) \neq (3, \mathbb{F}_4)$. Then $(C(\text{SL}_n(\mathbb{F})), \theta)$ is a good pair.
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Start with two arbitrary maximal simplices $\sigma_1$ and $\sigma_2$ in $C^\theta$.

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July 19, 2009 | TU Darmstadt | Max Horn | 18
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Theorem (H., van Maldeghem 2009)

Let $G$ be a group with 2-spherical $\mathbb{F}$-locally split root group datum, where $\text{char}\mathbb{F} \neq 2$ and $|\mathbb{F}| \geq 5$. Then $(C(G), \theta)$ is a good pair for any (twisted) Chevalley involution $\theta$ of $G$.

Proof by studying local case, i.e., involutions and polarities of Moufang planes, quadrangles and hexagons. Determine: $R_\theta$ connected? Direct ascent into $R_\theta$ possible?

Corollary

Let $G$ be a group with 2-spherical $\mathbb{F}$-locally split root group datum, where $\text{char}\mathbb{F} \neq 2$ and $|\mathbb{F}| \geq 5$. Then $C^\theta$ is pure and residually connected, hence geometric, for any (twisted) Chevalley involution $\theta$ of $G$. 
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Finding good pairs

Theorem (H., van Maldeghem 2009)

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On finitely generated unitary forms

In geometric group theory, so-called finiteness properties of groups are of high interest. (Examples: finite generation and finite presentation.)

Theorem (Gramlich, H., and Mühlherr, 2009)
Let \( G \) be a 2-spherical Kac-Moody group over a finite field \( \mathbb{F}_q \), \( q \) odd and \( \geq 5 \). Suppose \( \theta \) is an involutory automorphism which interchanges the two conjugacy classes of Borel subgroups. Then \( K := \text{Fix}_G(\theta) \) is finitely generated.

- Constant bound on \( q \), does not depend on the rank of \( G \).
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Isomorphisms of unitary forms of Kac-Moody groups over finite fields  
Unitary forms are finitely generated: Well, not always ...

Let $G$ be a non-spherical Kac-Moody group over $\mathbb{F}_{q^2}$ with unitary form $K$.

We have seen: if $G$ is 2-spherical and $q^2 > 4$, then $K$ is finitely generated.

If $G$ is not 2-spherical, then $K$ is not finitely generated, as observed recently by Caprace, Gramlich and Mühlherr.

- Let $T$ be a tree residue of the building. Then $G \cdot T$ is a simplicial tree (Dymara/Januszkiewicz).
- The key insight is the following: The action of the lattice $K$ on the simplicial tree $G \cdot T$ is minimal but ...
- ...there are infinitely many $K$-orbits on $G \cdot T$.
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Based on this evidence, one might conjecture: If $G$ is $(m+1)$-spherical, then $K$ is of type $F_m$ and “usually” the converse holds.
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