The geometry of involutions of algebraic groups and of Kac-Moody groups

TU Eindhoven
EIDMA Seminar Combinatorial Theory

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Overview

- Groups with a root datum
- Buildings
- Unitary forms
- Flip-flop systems and Phan geometries
- Properties and applications of flip-flop systems
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Chevalley groups: $SL_{n+1}$

Starting point: Chevalley groups. These are essentially determined by

1. a field $\mathbb{F}$ and
2. a (spherical) root system (more specifically, a root datum).

Root systems can be described and classified by Dynkin diagrams.

Example

$G = SL_{n+1}(\mathbb{F})$ corresponds to root system of type $A_n$ with this diagram:

![Dynkin diagram](image)

(This is also true for $PSL_{n+1}$; the notion of a root datum is needed to distinguish between them.)

For algebraically closed fields one obtains connected semi-simple linear algebraic groups; for finite fields (untwisted) finite groups of Lie type.
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For algebraically closed fields one obtains connected semi-simple linear algebraic groups; for finite fields (untwisted) finite groups of Lie type.
Let $n = 2$ and $G = \text{SL}_3(\mathbb{F})$. The associated root system $\Phi$ of type $A_2$:

\[
\begin{align*}
\beta & \quad \alpha + \beta \\
-\alpha & \quad \alpha \\
-\alpha - \beta & \quad -\beta
\end{align*}
\]

To each root $\rho \in \Phi$ a root group $U_\rho \cong (\mathbb{F}, +)$ of $G$ is associated:

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\begin{align*}
U_\alpha &= \left\langle \begin{pmatrix} 1 & * & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle, \\
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The root groups, the (commutator) relations between them and the torus $T := \bigcap_{\rho \in \Phi} N_G(U_\rho)$ (diagonal matrices in $G$) determine $G$ completely.
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Kac-Moody groups generalize Chevalley groups in a natural way. Again take . . .

1. a field $\mathbb{F}$ and

2. a root system (root datum) whose Dynkin diagram has edge labels in \{3, 4, 6, 8, $\infty$\}.

(Again: need root datum, not just root system, to distinguish $\text{SL}$ from $\text{PSL}$.)

Example

Let $\mathbb{F}[t, t^{-1}]$ denote the ring of Laurent polynomials over $\mathbb{F}$. $G = \text{SL}_{n+1}(\mathbb{F}[t, t^{-1}])$ is a Kac-Moody group over $\mathbb{F}$ with root system of type $\widetilde{\mathbb{A}}_n$:
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To obtain the root system of type \( \tilde{A}_n \) we add a new root corresponding to the lowest root in \( A_n \). For \( n = 3 \), we get a new root \( \gamma \) corresponding to \( -\alpha - \beta \).

The positive fundamental root groups now are:

\[
U_\alpha = \left\langle \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) \mid a \in \mathbb{F} \right\rangle, \quad U_\beta = \left\langle \left( \begin{array}{cc} 1 & 0 \\ 1 & a \end{array} \right) \mid a \in \mathbb{F} \right\rangle, \quad U_\gamma = \left\langle \left( \begin{array}{cc} 1 & 0 \\ at & 1 \end{array} \right) \mid a \in \mathbb{F} \right\rangle.
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The negative root groups can be obtained from the positive ones by applying the Chevalley involution of \( G \): Transpose, invert and swap \( t \) and \( t^{-1} \), hence

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\( G \) is generated by its root groups.

Important consequence: The groups \( U_+ = \left\langle U_\rho \mid \rho \in \Pi \right\rangle \) and \( U_- = \left\langle U_{-\rho} \mid \rho \in \Pi \right\rangle \) are no longer conjugate to each other.
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- Groups with a root datum
- Buildings
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What is a building

Buildings are...

▶ “geometries” for groups with root datum, such as algebraic and Kac-Moody groups and finite groups of Lie type.

Example: For \( G = \text{SL}_{n+1}(\mathbb{F}) \) the projective space

\[
\mathbb{P}^n(\mathbb{F}) = \{ U \subset \mathbb{F}^{n+1} \mid 0 \neq U \neq V \}.
\]

▶ isomorphic to the homogeneous space \( G/B \), where \( B = N_G(U) \) and \( U \) is generated by all positive (fundamental) root groups.

Example: For \( G = \text{SL}_{n+1}(\mathbb{F}) \),

▶ \( U \) is the group of unit upper triangular matrices and
▶ \( B \) is the group of upper triangular matrices.

▶ isomorphically to a simplicial complex, thus have a topological realization.

▶ are versatile, have many interpretations and countless applications.

Careful: One group may act on several buildings. Only the choice of a system of root groups resp. the group \( B \) determines the building.
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Some properties of buildings

Let $C$ be the building associated to a group $G$ with root group datum. Let $(W, S)$ be the Coxeter system with Coxeter diagram equal to that of $G$.

Some properties of $C$:

- Labeled simplicial complex, with labels from $S$ → every simplex has a type, a subset of $S$

- System $\mathcal{A}$ of subcomplexes called apartments, and isomorphic to the Coxeter complex of $(W, S)$

- Weyl-distance $\delta : C \times C \rightarrow W$ assigns distances to pairs of simplices

- Numerical distance $l : C \times C \rightarrow \mathbb{N}$ defined by $l(\sigma_1, \sigma_2) := l(\delta(\sigma_1, \sigma_2))$

- Building is called spherical if $l$ is bounded → notion of opposite simplices
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Let $\mathcal{C}$ be the building associated to a group $G$ with root group datum. Let $(\mathcal{W}, S)$ be the Coxeter system with Coxeter diagram equal to that of $G$.

Some properties of $\mathcal{C}$:

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- System $\mathcal{A}$ of subcomplexes called apartments, and isomorphic to the Coxeter complex of $(\mathcal{W}, S)$
- Weyl-distance $\delta : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{W}$ assigns distances to pairs of simplices
- Numerical distance $l : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{N}$ defined by $l(\sigma_1, \sigma_2) := l(\delta(\sigma_1, \sigma_2))$
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Unitary forms

- Let $G$ be Chevalley / Kac-Moody group over $\mathbb{F}$, and $\sigma \in \text{Aut}(\mathbb{F})$ with $\sigma^2 = \text{id}$.
- Let $\theta$ be the composition of the Chevalley involution of $G$ with $\sigma$. For $\text{SL}_n(\mathbb{F})$: $\theta : x \mapsto (\sigma(x)^T)^{-1}$.
- Then $K := \text{Fix}_G(\theta)$ is called (\sigma-)unitary form of $G$.

Examples

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\sigma$</th>
<th>$K$</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}_{n+1}(\mathbb{F})$</td>
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<tr>
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<td>$x \mapsto \bar{x}$</td>
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<td></td>
</tr>
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<td>$x \mapsto x^q$</td>
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- Let $G$ be Chevalley / Kac-Moody group over $\mathbb{F}$, and $\sigma \in \text{Aut}(\mathbb{F})$ with $\sigma^2 = \text{id}$.
- Let $\theta$ be the composition of the Chevalley involution of $G$ with $\sigma$. For $\text{SL}_n(\mathbb{F})$:
  \[
  \theta : x \mapsto (\sigma(x)^T)^{-1}.
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- Then $K := \text{Fix}_G(\theta)$ is called ($\sigma$-)unitary form of $G$.

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- Groups with a root datum
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Yes!

- A (twisted) Chevalley involution $\theta$ of $G$ induces building automorphism of $C$.
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For $\sigma \in \mathcal{C}$ the local flip-flop system $\mathcal{C}_\sigma^\theta$ consists of simplices in $\text{lk} \sigma$ for which the numerical $\theta$-distance is maximal among all simplices in the link.

We say $\mathcal{C}_\sigma^\theta$ allows direct descent if any simplex in $\text{lk} \sigma$ is connected to a simplex in $\mathcal{C}_\sigma^\theta$ by a minimal gallery along which $l^\theta$ is strictly increasing.

Call $(\mathcal{C}, \theta)$ a good pair if for all corank 2 simplices $\sigma$, $\mathcal{C}_\sigma^\theta$ is connected and allows direct descent.
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If $(C, \theta)$ is a good pair, then $C^\theta$ is path connected and pure (i.e., all its maximal simplices have equal type $J$ for some spherical subset $J$ of $S$). In fact $C^\theta$ is residually connected, hence geometric.

Example

Let $\theta$ be the twisted Chevalley involution of $SL_n(F)$, $F \neq F_4$. Then $(C(SL_n(F)), \theta)$ is a good pair. Therefore $C^\theta$ is geometric; we call the corresponding incidence geometry the flip-flop geometry or Phan geometry.
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Sketch of proof

Start with two arbitrary maximal simplices $\sigma_1$ and $\sigma_2$ in $C^\theta$.

- Choose maximal simplices $\bar{\sigma}_i$ in $C$, $i \in \{1, 2\}$, such that $\sigma_i$ is a face of $\bar{\sigma}_i$.

- Choose a minimal gallery $\gamma$ between $\bar{\sigma}_1$ and $\bar{\sigma}_2$ inside $C$.

- Using the condition on corank 2 simplices, transform $\gamma$ by bypassing chambers with low numerical $\theta$-distance, gradually increasing the maximal numerical $\theta$-distance of chambers in $\gamma$.

- Ultimately, num. $\theta$-distance is non-decreasing along $\gamma \rightarrow$ actually constant.

- Adjacent chambers with equal num. $\theta$-distance have equal $\theta$-distance
  $\implies \bar{\sigma}_1$ and $\bar{\sigma}_2$ have equal $\theta$-distance
  $\implies \sigma_1$ and $\sigma_2$ have same type and $C^\theta$ is connected.

- Finally, show that residual connectedness is inherited from $C$. 
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- Finally, show that residual connectedness is inherited from \( C \).
Start with two arbitrary maximal simplices $\sigma_1$ and $\sigma_2$ in $C^\theta$.

- Choose maximal simplices $\bar{\sigma}_i$ in $C$, $i \in \{1, 2\}$, such that $\sigma_i$ is a face of $\bar{\sigma}_i$.
- Choose a minimal gallery $\gamma$ between $\bar{\sigma}_1$ and $\bar{\sigma}_2$ inside $C$.
- Using the condition on corank 2 simplices, transform $\gamma$ by bypassing chambers with low numerical $\theta$-distance, gradually increasing the maximal numerical $\theta$-distance of chambers in $\gamma$.
- Ultimately, num. $\theta$-distance is non-decreasing along $\gamma \rightarrow$ actually constant.
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Finding good pairs

Theorem (H., van Maldeghem 2009)

Let $G$ be a group with 2-spherical $\mathbb{F}$-locally split root group datum, where $\text{char} \mathbb{F} \neq 2$ and $|\mathbb{F}| \geq 5$. Then $(C(G), \theta)$ is a good pair for any (twisted) Chevalley involution $\theta$ of $G$.

Proof by studying local case, i.e., involutions and polarities of Moufang planes, quadrangles and hexagons. Determine: $R_\theta$ connected? Direct descent into $R_\theta$ possible?

Corollary

Let $G$ be a group with 2-spherical $\mathbb{F}$-locally split root group datum, where $\text{char} \mathbb{F} \neq 2$ and $|\mathbb{F}| \geq 5$. Then $C^\theta$ is pure and residually connected, hence geometric, for any (twisted) Chevalley involution $\theta$ of $G$. 
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On finitely generated unitary forms

In geometric group theory, so-called finiteness properties of groups are of high interest. Among these are finite generation and finite presentation.

Theorem (Gramlich, H., and Mühlherr, 2008)

Let $G$ be a 2-spherical Kac-Moody group over a finite field $\mathbb{F}_q$, $q \geq 5$. Suppose $\theta$ is an involutory automorphism which interchanges the two conjugacy classes of Borel subgroups. If $q$ is odd then $K := \text{Fix}_G(\theta)$ is finitely generated.

- Constant bound on $q$, does not depend on the rank $n$
- Works for a large class of abstract involutory automorphisms
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Sketch of proof

1. Recall that $C^\theta$ is a subcomplex of the building $\Delta$ and $K.C^\theta \subseteq C^\theta$.

2. $C^\theta$ is pure and path connected since $G$ is $\mathbb{F}_q$-locally split and $q$ odd.

3. Choose a system $X'$ of representatives of the $K$-orbits on the maximal simplices in $C^\theta$. For each $\sigma \in X'$ pick a maximal simplex $\bar{\sigma} \in C$ containing $\sigma$. Set $X := \{\bar{\sigma} \mid \sigma \in X'\}$.

4. Since $C^\theta$ is connected, by standard arguments we have

$$K = \langle Stab_K(\sigma) \mid \sigma \text{ is a facet of } \sigma_0 \in X \rangle.$$ 

5. Show: $X'$ and hence $X$ is finite: Identify $K$-orbits on $C^\theta$ bijectively with orbits on a suitable maximal torus $T$. But here maximal tori are finite.

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*Involutions of Kac-Moody groups.*  

H.: Oberwolfach report

Ralf Gramlich and Andreas Mars.  
Isomorphisms of unitary forms of Kac-Moody groups over finite fields  
Unitary forms are finitely generated: Well, not always . . .

Let $G$ be a non-spherical Kac-Moody group over $\mathbb{F}_{q^2}$ with unitary form $K$. We have seen: if $G$ is 2-spherical and $q^2 > 4$, then $K$ is finitely generated.

If $G$ is not 2-spherical, then $K$ is not finitely generated, as observed recently by Caprace, Gramlich and Mühlherr.

- Let $T$ be a tree residue of the building. Then $G \cdot T$ is a simplicial tree (Dymara/Januszkiewicz).
- The key insight is the following: The action of the lattice $K$ on the simplicial tree $G \cdot T$ is minimal but . . .
- . . . there are infinitely many $K$-orbits on $G \cdot T$.
- It follows (Bass) that the lattice $K$ cannot be finitely generated.

Based on this evidence, one might conjecture: If $G$ is $(m + 1)$-spherical, then $K$ is of type $F_m$ and “usually” the converse holds.
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