From Fischer spaces to (Lie) algebras

Max Horn

joint work with
H. Cuypers, J. in ’t panhuis, S. Shpectorov

Technische Universität Braunschweig

Buildings 2010
Overview

1. 3-transposition groups and Fischer spaces
2. Algebras from Fischer spaces
3. Vanishing sets
4. Lie algebras
5. Some computations
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3-transposition groups

A class of 3-transpositions in a group $G$ is a conjugacy class $D$ of $G$ such that

1. the elements of $D$ are involutions and
2. for all $d, e \in D$ the order of $de$ is equal to 1, 2 or 3.

$G$ is called 3-transposition group if $G = \langle D \rangle$.

Examples

- Transpositions in $G = \text{Sym}(n)$; $D = (12)^G$
- Transvections in $G = \text{U}(n,2)$; $D = d^G$ where
  
  $$d = \begin{pmatrix}
  0 & 0 & \ldots & 0 & 1 \\
  0 & 1 & \ldots & 0 & 0 \\
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  0 & 0 & \ldots & 1 & 0 \\
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  \end{pmatrix}$$
  (in GAP’s version of this group)
- $\text{Fi}_{22}$, $\text{Fi}_{23}$, $\text{Fi}_{24}$ (note: the simple group is $\text{Fi}'_{24}$)
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Classify 3-transposition groups

Fischer (around 1970) classified finite 3-transposition groups with no non-trivial normal solvable subgroups. 

Cuypers and Hall (90s) classified all (possibly infinite) 3-transposition groups with trivial center, using geometric methods (Fischer spaces).

Cuypers and Hall: If center is non-trivial, then $G/Z(G)$ is 3-transposition group with trivial center.
Fischer (around 1970) classified finite 3-transposition groups with no non-trivial normal solvable subgroups. This is equivalent to the classification of finite simple groups.

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Cuypers and Hall: If center is non-trivial, then $G/Z(G)$ is 3-transposition group with trivial center.
Throughout the rest of this talk, let $D$ be a class of 3-transpositions generating a 3-transposition group $G$, and $Z(G) = 1$.

- $o(de) = 3 \iff de \neq ed \iff d \neq d^e = e^d \neq e$

- The Fischer space $\Pi(D)$ is the partial linear space with $D$ as point set, and the triples $\{d, e, d^e\}$ as lines (when $o(de) = 3$).
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The Fischer space $\Pi(D)$ is the partial linear space with $D$ as point set, and the triples $\{d, e, d^e\}$ as lines (when $o(de) = 3$).
Proposition (Buekenhout)

A partial linear space is a Fischer space if and only if every pair of intersecting lines generates a subspace isomorphic to the dual of an affine plane of order 2, or an affine plane of order 3.
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Denote by $\mathbb{F}_2 D$ the $\mathbb{F}_2$ vector space with basis $D$.

- Vectors are finite subsets of $D$; sum of two sets is their symmetric difference.

- Define the 3-transposition algebra $\mathcal{A}(D)$ with underlying vector space $\mathbb{F}_2 D$; multiplication is linear expansion of multiplication defined on $d, e \in D$ by

\[
 d \ast e := \begin{cases} 
 d + e + e^d = \{d, e, e^d\} & \text{if } o(de) = 3 \\
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- $\mathcal{A}(D)$ is a non-associative commutative algebra.
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Group action on $\mathcal{A}(D)$

- $G$ acts on $\mathcal{A}(D)$ by conjugation:
  $$(d_1 + \ldots + d_n)^g = d_1^g + \ldots + d_n^g.$$ 

- Let $V$ be a subset of $\mathcal{A}(D)$. Then $I(V)$ denotes the ideal of $\mathcal{A}(D)$ generated by $V$.

- Goal: Compute Lie algebra quotients with a $G$-action.

- $G = \langle D \rangle$, so $I(V)$ is $G$-invariant if and only if for all $d \in D$, $X \in I(V)$ we have $X^d \in I(V)$. 

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**Lemma**

Let \( X \) be a finite subset of \( D \) and \( d \in D \). Then

\[
d \ast X = X + X^d + (|A_d \cap X| \mod 2)d.
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- \( X \) is called **vanishing set** if \( |A_d \cap X| \) is even for all \( d \in D \).
- Examples: Empty set; point sets of finite maximal linear subspaces of \( \Pi(D) \) (they have odd size); …
- **Vanishing ideals** are ideals generated by vanishing sets.
- Any vanishing ideal \( I \) is \( G \)-invariant: If \( X \in I \) then \( d \ast X \in I \) hence \( \{X^d \mid d \in D\} \in I \).
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The maximal vanishing ideal

Let $\mathcal{V}$ be the ideal of $\mathcal{A}$ generated by all vanishing subsets of $D$.

**Lemma**

1. $\mathcal{V}$ equals the linear span of all vanishing subsets of $D$.
2. $\mathcal{V}$ is a proper ideal.

**Idea of proof:**

1. Follows from the fact that $\mathcal{V}$ is $G$-invariant.
2. Define a symplectic form $\langle \cdot | \cdot \rangle$ on $\mathcal{A}(D)$: Set $\langle d | e \rangle = 1$ if $de \neq ed$ and 0 otherwise; extend linearly. This form is non-zero if there are lines (i.e. if $G$ is non-abelian).

If $X$ is a vanishing set, then $\langle d | X \rangle = 0$. So $\mathcal{V}$ is in the radical of $\langle \cdot | \cdot \rangle$ and hence a proper subspace of $\mathcal{A}(D)$. 
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Proper $G$-invariant ideals are vanishing

**Lemma**

*Any $G$-invariant proper ideal of $\mathcal{A}(D)$ is contained in $\mathcal{V}.*

**Proof:**

Assume a $G$-invariant ideal $I$ containing a non-vanishing set $X$. There is $d \in D$ such that $d \ast X = d + X + X^d \in I$. But $I$ is $G$-invariant, thus $X^d \in I$ and so $d \in I$ and $d^G = D \subseteq I = \mathcal{A}$.

**Proposition**

*Suppose $Q$ is a simple quotient algebra of $\mathcal{A}(D)$. If $G$ induces a group of automorphisms on $Q$, then $Q$ is isomorphic to $\mathcal{A}(D)/\mathcal{V}.*
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When is (a quotient of) $A(D)$ a Lie algebra?

**Lemma**

Let $I$ be an ideal of $A(D)$. Then $A(D)/I$ is a Lie algebra, if and only if every affine plane $\pi$ of $\Pi(D)$ is in $I$.

If there are no affine planes, then $A(D)$ is a Lie algebra and $A(D)/\mathcal{V}$ is an abelian Lie algebra.

If affine planes are not vanishing sets, then no non-trivial quotient of $A(D)$ is a Lie algebra.
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Simple Lie algebras from 3-transposition groups

Theorem

Let $D$ be a class of 3-transpositions generating a finite group $G$ satisfying a certain irreducibility condition. Suppose $\mathcal{A}(D)/\mathcal{V}$ is a simple Lie algebra over $\mathbb{F}_2$ of dimension at least 2.

Then $\mathcal{A}(D)/\mathcal{V}$ is isomorphic to one of the following:

1. $^2A_n(2)$ if $G = 3^n : W(A_n)$ or $SU_{n+1}(2)$; for $n = 5$ also $P\Omega^-_6(3)$.
2. $^2D_n(2)$ if $G = 3^n : W(D_n)$ and $n$ odd.
3. $D_n(2)$ if $G = 3^n : W(D_n)$ and $n$ even; for $n = 4$ also $P\Omega^+_8(2) : Sym_3$.
4. $^2E_6(2)$ if $G = 3^6 : W(E_6)$ or $P\Omega_7(3)$ or $Fi_{22}$.
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2. $^2D_n(2)$ if $G = 3^n : W(D_n)$ and $n$ odd.
3. $D_n(2)$ if $G = 3^n : W(D_n)$ and $n$ even; for $n = 4$ also $P\Omega^+_8(2) : Sym_3$.
4. $^2E_6(2)$ if $G = 3^6 : W(E_6)$ or $P\Omega_7(3)$ or $Fi_{22}$.
5. $E_7(2)$, $E_8(2)$ if $G = 3^n : W(E_n)$. 
Simple Lie algebras from 3-transposition groups

Theorem

Let $D$ be a class of 3-transpositions generating a finite group $G$ satisfying a certain irreducibility condition. Suppose $A(D)/\mathcal{V}$ is a simple Lie algebra over $\mathbb{F}_2$ of dimension at least 2.

Then $A(D)/\mathcal{V}$ is isomorphic to one of the following:

1. $^2A_n(2)$ if $G = 3^n : W(A_n)$ or $SU_{n+1}(2)$; for $n = 5$ also $P\Omega_6^{-}(3)$.
2. $^2D_n(2)$ if $G = 3^n : W(D_n)$ and $n$ odd.
3. $D_n(2)$ if $G = 3^n : W(D_n)$ and $n$ even; for $n = 4$ also $P\Omega_8^{+}(2) : \text{Sym}_3$.
4. $^2E_6(2)$ if $G = 3^6 : W(E_6)$ or $P\Omega_7(3)$ or $Fi_{22}$.
5. $E_7(2), E_8(2)$ if $G = 3^n : W(E_n)$.  

Vanishing sets

Lie algebras

Some computations
Overview

1. 3-transposition groups and Fischer spaces
2. Algebras from Fischer spaces
3. Vanishing sets
4. Lie algebras
5. Some computations
**Unitary groups**

$L_{\text{max}}$ denotes the maximal Lie algebra quotient of $\mathcal{A}(D)$.

| $G$   | $|D|$  | dim $L_{\text{max}}$ | dim $\mathcal{A}(D)/\mathcal{V}$ |
|-------|--------|-----------------------|----------------------------------|
| $U_2(2)$ | 3      | 3                     | 2                               |
| $U_3(2)$ | 9      | 8                     | 8                               |
| $U_4(2)$ | 45     | 30                    | 14                              |
| $U_5(2)$ | 165    | 45                    | 24                              |
| $U_6(2)$ | 693    | 78                    | 34                              |
| $U_7(2)$ | 2709   | 119                   | 48                              |
| $U_8(2)$ | 10789  | 176                   | 62                              |
| $U_9(2)$ | 43356  | 249                   | 80                              |
| $U_{10}(2)$ | 174933 | 340                   | 98                              |
| $U_{11}(2)$ | ?      | ?                     | 120                             |

$U_n(2) = \frac{1}{6}(4^n + (-2)^n - 2)$

In fact, $\mathcal{A}(D)/\mathcal{V} \cong ^2A_n(2)$ holds.
Sporadic cases

$L_{\text{max}}$ denotes the maximal Lie algebra quotient of $A(D)$.

| $G$               | $|D|$ | $\text{dim } L_{\text{max}}$ | $\text{dim } A(D)/\mathcal{V}$ |
|-------------------|------|-------------------------------|-------------------------------|
| $O^+(8, 2) : \text{Sym}_3$ | 360  | 52                           | 26                           |
| $O^+(8, 3) : \text{Sym}_3$ | 3240 | 0                            | 782                          |
| $\text{Fi}_{22}$            | 3510 | 78                           | 78                           |
| $\text{Fi}_{23}$            | 31671| 0                            | 782                          |
| $\text{Fi}_{24}$            | 306936| 0                           | 3774                         |

For $O^+(8, 2) : \text{Sym}_3$ we get the simple Lie algebra $D_4(2)$ and for $\text{Fi}_{22}$ the simple Lie algebra $^2E_6(2)$.

In the other cases, we do not get Lie algebras, but still a non-trivial algebra structure.
Thank you!