The geometry of involutions of algebraic groups and of Kac-Moody groups

International Workshop on Algebraic Groups,
Quantum Groups and Related Topics

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Overview

TECHNISCHE UNIVERSITÄT DARMSTADT

- Groups with a root datum
- Buildings
- Unitary forms
- Flip-flop systems and Phan geometries
- Properties and applications of flip-flop systems

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Chevalley groups: SL_{n+1}

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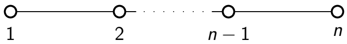
Starting point: Chevalley groups. These are essentially determined by

1. a field $\mathbb{F}$ and
2. a (spherical) root system (more specifically, a root datum).

Root systems can be described and classified by Dynkin diagrams.

Example

$G = \mathrm{SL}_{n+1}(\mathbb{F})$ corresponds to root system of type A_n with this diagram:



(Also true for PSL_{n+1}; one needs a root datum to distinguish between them.)

For algebraically closed fields one obtains connected semi-simple linear algebraic groups; for finite fields (untwisted) finite groups of Lie type.

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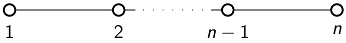

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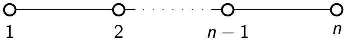

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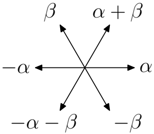
SL_3 as an example; root groups

Let $n = 2$ and $G = \mathsf{SL}_3(\mathbb{F})$. The associated root system Φ of type A_2:



To each root $\rho \in \Phi$ a root group $U_\rho \cong (\mathbb{F}, +)$ of G is associated:

$$U_\alpha = \left\langle \begin{pmatrix} 1 & * & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\rangle, U_\beta = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ & 1 & * \\ & & 1 \end{pmatrix} \right\rangle, U_{\alpha+\beta} = \left\langle \begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\rangle, U_{-\alpha} = (U_\alpha^T)^{-1}, \ldots$$

The root groups, the (commutator) relations between them and the torus $T := \bigcap_{\rho\in\Phi} N_G(U_\rho)$ (diagonal matrices in G) determine G completely.

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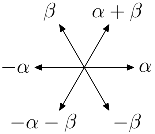

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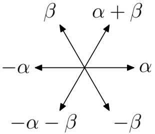

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Kac-Moody groups

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Kac-Moody groups generalize Chevalley groups in a natural way. Again take ...

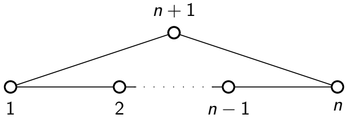
1. a field $\mathbb{F}$ and
2. a root system (root datum) whose Dynkin diagram has edge labels in $\{3, 4, 6, 8, \infty\}$.

(Again: need root datum, not just root system, to distinguish SL from PSL.)

Example

Let $\mathbb{F}[t, t^{-1}]$ denote the ring of Laurent polynomials over $\mathbb{F}$.

$G = \mathrm{SL}_{n+1}(\mathbb{F}[t, t^{-1}])$ is a Kac-Moody group over $\mathbb{F}$ with root system of type $\widetilde{A_n}$:



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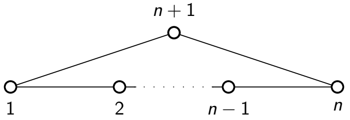

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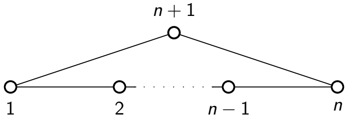

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Root groups in Kac-Moody groups

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To obtain the root system of type $\widetilde{A}_n$ we add a new root corresponding to the lowest root in A_n. For $n = 2$, we get a new root γ corresponding to $-\alpha - \beta$.

The positive fundamental root groups now are:

$$U_\alpha = \left\langle \begin{pmatrix} 1 & a & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid a \in \mathbb{F} \right\rangle, U_\beta = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ & 1 & a \\ & & 1 \end{pmatrix} \mid a \in \mathbb{F} \right\rangle, U_\gamma = \left\langle \begin{pmatrix} 1 & & \\ 0 & 1 & \\ at & 0 & 1 \end{pmatrix} \mid a \in \mathbb{F} \right\rangle.$$

The negative root groups can be obtained from the positive ones by applying the Chevalley-Cartan involution of G: Transpose, invert and swap t and t^{-1}, hence

$$U_{-\gamma} = \left\langle \begin{pmatrix} 1 & 0 & -at^{-1} \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid a \in \mathbb{F} \right\rangle \quad \text{and } U_\alpha, U_\beta \text{ as before.}$$

G is generated by its root groups.

Important consequence: The groups $U_+ = \langle U_\rho \mid \rho \in \Phi_+ \rangle$ and $U_- = \langle U_\rho \mid \rho \in \Phi_- \rangle$ are no longer conjugate to each other.

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- Groups with a root datum
- Buildings
- Unitary forms
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What is a building

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Let G be a group with root datum.

The building $\mathcal{C}(G)$ of G can be realized as ...

- ...a homogeneous space G/B, where $B = N_G(U)$ and U is generated by all positive root groups.

  Example: For $G = \mathrm{SL}_{n+1}(\mathbb{F})$,

  - U is the group of unit upper triangular matrices and
  - B is the group of upper triangular matrices.

- ...CAT(0)-spaces, an incidence geometry, a Chamber system, ...

- ...a simplicial complex: Take as simplices all proper subgroups of G containing B, ordered by reverse inclusion.

Careful: One group may act on several buildings. But the choice of a system of root groups resp. the group B determines the building.

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Some properties of buildings

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Leg G be a group with root datum, denote by $\mathcal{C} = \mathcal{C}(G)$ its associated building and by (W, S) its Coxeter system.

Some properties of $\mathcal{C}$:

- Labeled simplicial complex, with labels from $S \to$ every simplex has a type.
- System $\mathcal{A}$ of subcomplexes called apartments, each isomorphic to the Coxeter complex of (W, S). Any two simplices are contained in at least one apartment.
- Weyl-distance $\delta : \mathcal{C} \times \mathcal{C} \to W$ assigns "distances" to pairs of simplices.
- numerical distance $l : \mathcal{C} \times \mathcal{C} \to \mathbb{N}$ defined by $l(\sigma_1, \sigma_2) := l(\delta(\sigma_1, \sigma_2))$.
- Building is called spherical if l is bounded $\to$ notion of opposite simplices.

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Unitary forms

TECHNISCHE UNIVERSITÄT DARMSTADT

- Let G be Chevalley / Kac-Moody group over $\mathbb{F}$, and $\sigma \in \mathrm{Aut}(\mathbb{F})$ with $\sigma^2 = \mathrm{id}$.
- Let θ be the composition of the Chevalley-Cartan involution of G with σ. For $\mathrm{SL}_n(\mathbb{F})$:

$$\theta : x \mapsto (\sigma(x)^T)^{-1}.$$

- Then $K := \mathrm{Fix}_G(\theta)$ is called (σ-)unitary form of G.

Examples

G	σ	K	Remark
$\mathrm{SL}_{n+1}(\mathbb{F})$	$\mathrm{id}_{\mathbb{F}}$	$\mathrm{SO}_{n+1}(\mathbb{F})$	
$\mathrm{SL}_{n+1}(\mathbb{C})$	$x \mapsto \bar{x}$	$\mathrm{SU}_{n+1}(\mathbb{R})$	defined over $\mathbb{C}$; $\mathbb{R}$-form of G
$\mathrm{SL}_{n+1}(\mathbb{F}_{q^2})$	$x \mapsto x^q$	$\mathrm{SU}_{n+1}(\mathbb{F}_q)$	defined over $\mathbb{F}_{q^2}$
$\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$	$x \mapsto x^q$	$\mathrm{Sp}_{2n}(\mathbb{F}_q)$	
$\mathrm{SL}_{n+1}(\mathbb{F}_{q^2}[t, t^{-1}])$	$x \mapsto x^q$	$\mathrm{SU}_{n+1}(X)$	$X = \langle \lambda \cdot (t + \varepsilon t^{-1}) \mid \varepsilon = \pm 1, \lambda \in \mathbb{F}_{q^2}, \sigma(\lambda) = \varepsilon\lambda \rangle$

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Geometries for unitary forms

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Applications

- Phan type theorems (Bennett, Devillers, Gramlich, Hoffman, H., Mühlherr, Nickel, Shpectorov)
- New lattices in Kac-Moody groups (Gramlich, Mühlherr)
- Automorphisms of unitary forms of Kac-Moody groups (Kac, Peterson; Caprace; Gramlich, Mars)
- Representation theory (Devillers, Gramlich, Mühlherr, Witzel): Generalize Solomon-Tits theorem
- Generalized Iwasawa decomposition (De Medts, Gramlich, H.): G split conn. reductive $\mathbb{F}$-group / Kac-Moody group over $\mathbb{F}$. When does $G_\mathbb{F}$ admit a decomposition $G_\mathbb{F} = K_\mathbb{F} B_\mathbb{F}$ (where K is centralizer of an involution)? (Inspired by Helminck & Wang, 1993.)
- Finiteness properties (Caprace, Devillers, Gramlich, H., Mühlherr, Witzel)

Structure of flip-flop systems: Good pairs

TECHNISCHE UNIVERSITÄT DARMSTADT

- Let θ be an involutory almost-isometry of a building $\mathcal{C}$.
- For $\sigma \in \mathcal{C}$ the local flip-flop system $\mathcal{C}_\sigma^\theta$ consists of simplices in $\operatorname{lk} \sigma$ for which the numerical θ-distance is maximal among all simplices *in the link*.
- Call $(\mathcal{C}, \theta)$ a good pair if for all corank-2 simplices $\sigma \in \mathcal{C}$, $\mathcal{C}_\sigma^\theta$ is path connected and "allows direct ascent".

Theorem (Gramlich, H., Mühlherr 2008)

If $(\mathcal{C}, \theta)$ is a good pair, then $\mathcal{C}^\theta$ is path connected and pure, i.e., all its maximal simplices have equal type $J \subset S$. Moreover $\mathcal{C}^\theta$ is residually connected, hence there exists an associated incidence geometry, the Phan geometry.

Example (Bennet, Shpectorov)

Let θ be a twisted Chevalley involution of $\mathrm{SL}_n(\mathbb{F})$, $n \geq 3$ and $(n, \mathbb{F}) \neq (3, \mathbb{F}_4)$. Then $(\mathcal{C}(\mathrm{SL}_n(\mathbb{F})), \theta)$ is a good pair.

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Structure of flip-flop systems: Sketch of proof

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Start with two arbitrary maximal simplices σ_1 and σ_2 in $\mathcal{C}^\theta$.

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- Consider $\mathcal{C}^\theta_\sigma$ for corank-2 simplices σ in γ. Using our conditions on these, transform γ by "bypassing" chambers with low numerical θ-distance, gradually increasing the maximal numerical θ-distance of chambers in γ.
- Ultimately, num. θ-distance is non-decreasing along γ $\rightarrow$ actually constant.
- Show: Adjacent chambers with equal num. θ-distance have equal θ-distance
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Finding good pairs

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Theorem (H., van Maldeghem 2009)

Let G be a group with 2-spherical $\mathbb{F}$-locally split root group datum, where $\operatorname{char}\mathbb{F} \neq 2$ *and* $|\mathbb{F}| \geq 5$*. Then* $(\mathcal{C}(G), \theta)$ *is a good pair for any (twisted) Chevalley involution θ of G.*

Proof by studying local case, i.e., involutions and polarities of Moufang planes, quadrangles and hexagons. Determine: R_θ connected? Direct ascent into R_θ possible?

Corollary

Let G be a group with 2-spherical $\mathbb{F}$-locally split root group datum, where $\operatorname{char}\mathbb{F} \neq 2$ *and* $|\mathbb{F}| \geq 5$*. Then C^θ is pure and residually connected, hence geometric, for any (twisted) Chevalley involution θ of G.*

Finding good pairs

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Corollary

Let G be a group with 2-spherical $\mathbb{F}$-locally split root group datum, where $\mathrm{char}\mathbb{F} \neq 2$ and $|\mathbb{F}| \geq 5$. Then $\mathcal{C}^\theta$ is pure and residually connected, hence geometric, for any (twisted) Chevalley involution θ of G.

On finitely generated unitary forms

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In geometric group theory, so-called finiteness properties of groups are of high interest. (Examples: finite generation and finite presentation.)

Theorem (Gramlich, H., and Mühlherr, 2009)

Let G be a 2-spherical Kac-Moody group over a finite field $\mathbb{F}_q$, q odd and ≥ 5. Suppose θ is an involutory automorphism which interchanges the two conjugacy classes of Borel subgroups. Then $K := \mathrm{Fix}_G(\theta)$ is finitely generated.

- Constant bound on q, does not depend on the rank of G.
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Recall that $\mathcal{C}^\theta$ is a subcomplex of the building Δ and K acts on it.

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2. Denote by $\overline{\mathcal{C}^\theta}$ union of $\mathcal{C}^\theta$ with stars in $\mathcal{C}$ of all maximal simplices of $\mathcal{C}^\theta$.
3. K acts on maximal simplices in $\overline{\mathcal{C}^\theta}$. Assume there is only a single K-orbit.
4. $\mathcal{C}^\theta$ is connected $\iff$ $\overline{\mathcal{C}^\theta}$ is connected. Pick a maximal simplex $\sigma_0 \in \overline{\mathcal{C}^\theta}$:
$$K = \langle Stab_K(\sigma) \mid \sigma \text{ is a facet of } \sigma_0 \rangle \,.$$
5. Caprace and Mühlherr: Stabilizers in K of corank-1 simplices of $\mathcal{C}$ are finite.
6. In general, show that there are only finitely many K-orbits on $\overline{\mathcal{C}^\theta}$: Identify them bijectively with orbits on a maximal θ-split torus T of G acting on itself via $(t_1, t_2) \mapsto \theta(t_2)^{-1} t_1 t_2$. But here maximal tori are finite.
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Unitary forms are finitely generated: Well, not always . . .

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Let G be a non-spherical Kac-Moody group over $\mathbb{F}_{q^2}$ with unitary form K.

We have seen: if G is 2-spherical and $q^2 > 4$, then K is finitely generated.

If G is *not* 2-spherical, then K is not finitely generated, as observed recently by Caprace, Gramlich and Mühlherr.

- Let T be a tree residue of the building. Then $G.T$ is a simplicial tree (Dymara/Januszkiewicz).
- The key insight is the following: The action of the lattice K on the simplicial tree $G.T$ is minimal but . . .
- . . . there are infinitely many K-orbits on $G.T$.
- It follows (Bass) that the lattice K cannot be finitely generated.

Based on this evidence, one might conjecture: If G is $(m+1)$-spherical, then K is of type F_m and "usually" the converse holds.

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