Unitary forms of Kac-Moody groups

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Overview

- Finite groups of Lie type
- Kac-Moody groups over finite fields
- Unitary forms
- Geometry and group theory
- Phan theory: Presentations of groups
- Finiteness properties

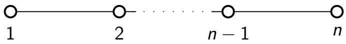
Finite groups of Lie type

Starting point: (untwisted) finite groups of Lie type. These are essentially determined by

1. a (finite) field $\mathbb{F}_q$ and
2. a (spherical) root system (more specifically, a root datum).

Example

$G = \mathrm{SL}_{n+1}(\mathbb{F}_q)$ corresponds to the root system of type A_n with this Coxeter diagram:



(This is also true for PSL_{n+1} und GL_{n+1}; the notion of a root datum is needed to distinguish between them.)

SL_3 as an example; root groups

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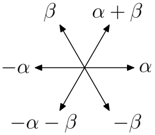
Let $n = 2$ and $G = SL_3(\mathbb{K})$. The associated root system Φ of type A_2:



To each root $\rho \in \Phi$ a root group $U_\rho \cong (\mathbb{K}, +)$ of G is associated:

$$U_\alpha = \left\langle \begin{pmatrix} 1 & * & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\rangle, U_\beta = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ & 1 & * \\ & & 1 \end{pmatrix} \right\rangle, U_{\alpha+\beta} = \left\langle \begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\rangle, U_{-\alpha} = {}^T U_\alpha^{-1}, \ldots$$

The root groups, the (commutator) relations between them and the torus $T := \bigcap_{\rho \in \Phi} N_G(U_\rho)$ (diagonal matrices in G) determine G completely.

Rank 1 and rank 2 subgroups

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Let G be an (untwisted) finite group of Lie type with root system Φ. Let Π be a fundamental system of Φ.

For $\alpha \in \Pi$ we call $G_\alpha := \langle U_\alpha, U_{-\alpha} \rangle$ a rank 1 subgroup.

For $\alpha, \beta \in \Pi$ with $\beta \neq \pm\alpha$ we call $G_{\alpha\beta} := \langle G_\alpha, G_\beta \rangle$ a rank 2 subgroup.

Example

Let $G = \mathrm{SL}_{n+1}$.

- rank 1 subgroups: block diagonal SL_2s
- rank 2 subgroups: block diagonal SL_3s or $(\mathrm{SL}_2 \times \mathrm{SL}_2)$s

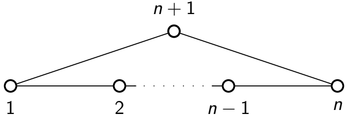
Kac-Moody groups over finite fields

(Split) Kac-Moody groups over finite fields generalize (untwisted) finite groups of Lie type in a natural way. Take the following ingredients:

1. a (finite) field $\mathbb{K}$ and
2. a root system (root datum) whose Coxeter diagram has edge labels in $\{3, 4, 6, 8, \infty\}$.

Example

$G = \mathrm{SL}_{n+1}(\mathbb{F}_q[t, t^{-1}])$ is a Kac-Moody group over $\mathbb{F}_q$ with root system of type $\widetilde{A}_n$:



($\mathbb{F}_q[t, t^{-1}]$ is the ring of Laurent polynomials over $\mathbb{F}_q$.)

Again: need root data to distinguish SL from PSL and GL.

Root groups in Kac-Moody groups

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To obtain the root system of type $\widetilde{A}_n$ we add a new root corresponding to the lowest root in A_n. For $n = 3$, we get a new root γ corresponding to $-\alpha - \beta$.

The positive fundamental root groups now are the following:

$$U_\alpha = \left\langle \begin{pmatrix} 1 & a & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\rangle, U_\beta = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ & 1 & a \\ & & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\rangle, U_\gamma = \left\langle \begin{pmatrix} 1 & & \\ 0 & 1 & \\ at & 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\rangle.$$

The negative root groups can be obtained from the positive ones by applying the Chevalley involution of G: Transpose, invert and swap t and t^{-1}, hence

$$U_{-\gamma} = \left\langle \begin{pmatrix} 1 & 0 & -at^{-1} \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\rangle.$$

G is generated by its root groups.

Unitary forms

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- Let G be a Kac-Moody group over $\mathbb{F}_{q^2}$.
- Let θ be the composition of the Chevalley involution of G with the field involution σ of $\mathbb{F}_{q^2}$. For matrix groups:

$$\theta : x \mapsto (\sigma(x)^T)^{-1}.$$

- Then $K := \mathrm{Fix}_G(\theta)$ is called unitary form of G.

Examples

- $G = \mathrm{SL}_{n+1}(\mathbb{F}_{q^2})$, then $K = \mathrm{SU}_{n+1}(\mathbb{F}_q)$.
- $G = \mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$, then $K = \mathrm{Sp}_{2n}(\mathbb{F}_q)$.
- $G = \mathrm{SL}_{n+1}(\mathbb{F}_{q^2}[t, t^{-1}])$, then $K = ...$

Geometry: buildings

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Buildings are ...

- ... geometries for algebraic, Kac-Moody, Lie type and other groups.

  Example: The projective space $\mathbb{P}^n(\mathbb{K})$ for $G = \mathrm{SL}_{n+1}(\mathbb{K})$.
- ... isomorphic to a simplicial complex, thus have topological realization.
- ... isomorphic to the homogeneous space G/B, where $B = N_G(U)$ and U is generated by all positive (fundamental) root groups.

  Example: For $G = \mathrm{SL}_{n+1}(\mathbb{K})$,
  - U is the group of unit upper triangular matrices and
  - B is the group of upper triangular matrices.
- ... are versatile and can be interpreted in many ways (chamber systems, $CAT(0)$-spaces, ...)

Careful: One group may act on several buildings. Only the choice of a system of root groups resp. the group B determines the building.

Why are buildings useful?

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They further our understanding of their groups.

- Each automorphism of a connected reductive algebraic or Kac-Moody group of rank at least 2 is induced by an automorphism of the building (Tits; Caprace-Mühlherr).
- Analogously for the automorphisms of the unitary forms of Kac-Moody groups (Kac-Peterson; Caprace; Gramlich-Mars).
- Representation theory: For algebraic and Lie type groups the building G/B is a wedge of spheres and the Steinberg representation is obtained by the action of G on the highest non-trivial homology group of G/B (Solomon-Tits).
- ... more in the following

Borel groups and automorphisms

Let Δ_+ be a building of a finite group of Lie type G, viewed as a simplicial complex.

- Then the Borel subgroup B (recall $B = N_G(U)$ where U is generated by all positive root groups) is the stabilizer of a maximal simplex in Δ.
- Thus G/B is isomorphic to the set of all maximal chambers in Δ. The simplicial complex can be reconstructed from this.
- This allows passage from group automorphisms to building automorphisms: If θ maps B to a conjugate of B, this induces an isometry of the building.
- In fact, every automorphism of G has this property.

Tits' lemma

Theorem (Tits' lemma)

Let G be a group acting transitively on a simplicial complex Δ, let σ be a maximal simplex in Δ. Then Δ is simply connected if and only if G is presented by the generators and relations contained in the stabilizers of non-empty faces of σ.

Example

- $G = \mathrm{SL}_{n+1}(\mathbb{K})$, $\Delta = \mathbb{P}^n(\mathbb{K})$
- G acts transitively on its building Δ (if $\mathbb{K} \neq \mathbb{F}_2$), which is simply connected.
- maximal simplex: the flag $\langle e_1 \rangle, \langle e_1, e_2 \rangle, \ldots, \langle e_1, \ldots, e_n \rangle$

Phan type theorems

Theorem

Let G be a finite group of Lie type over $\mathbb{F}_{q^2}$ and let K be its unitary form. If q is sufficiently large, then the relations contained in the rank 2 subgroups

$$K_{\alpha\beta} := G_{\alpha\beta} \cap K$$

are sufficient for a presentation of G by generators and relations.

Example

- $G = \mathrm{SL}_{n+1}(\mathbb{F}_{q^2})$, $K = \mathrm{SU}_{n+1}(\mathbb{F}_q)$, type A_n
- rank 1 subgroups: block diagonal SU_2s
- rank 2 subgroups: block diagonal SU_3s resp. $(\mathrm{SU}_2 \times \mathrm{SU}_2)$s

Ingredient of the (revised) classification of finite simple groups: Used to "recognize" groups from a system of known subgroups.

Phan type theorems: History of the proof(s)

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- Original proof: Computations in presentations.

  A_n, D_n, E_n Phan (1977)

- Phan program as part of the Gorenstein-Lyons-Solomon project:

  Define suitable subgeometry $\mathcal{C}^\theta$ of $\Delta(G)$ on which K acts transitively. Show that $\mathcal{C}^\theta$ is simply connected. Apply Tits' lemma. Finally, need to classify certain subgroup amalgams.

  A_n, B_n, C_n, D_n Bennett, Gramlich, Hoffman, Shpectorov (2003-2007)

  E_n, F_4 Devillers, Gramlich, Hoffman, Mühlherr, Shpectorov (2005-2008)

  Small cases Gramlich, H., Nickel (2005-2007)

  - A_3/D_3, $q = 3$: 9-fold (universal) cover exists
  - B_3, $q = 3$: 3^7-fold (universal) cover exists
  - B_3, $q \in \{5, 7, 8\}$; C_3, $q \in \{3, 4, 5, 7\}$; C_4, $q = 2$: Phan type theorem holds

Finiteness properties of G

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Let G be a Kac-Moody group over $\mathbb{F}_{q^2}$.

Since G is generated by its fundamental root subgroups, it is finitely generated (finiteness length ≥ 1).

Abramenko-Mühlherr (1997): If G is 2-spherical (all rank 2 subgroups are finite; more generally, no ∞ in the Coxeter diagram) and $q \geq 4$, then G is even finitely presented (finiteness length ≥ 2).

Open problem: If G is m-spherical, is the finiteness length $\geq m$? What about the converse?

Which finiteness properties does the unitary form K possess?

Finiteness properties of K

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Let G be a non-spherical Kac-Moody group over $\mathbb{F}_{q^2}$ with twin building Δ and unitary form K.

Theorem (Gramlich, Mühlherr)

If q is sufficiently large, then K is a lattice (discrete subgroup with finite covolume) in $Isom(\Delta)$, the (locally compact) group of all isometries of Δ.

Corollary

If $q^2 > \frac{1}{25}1764^n$ and G is 2-spherical, then K is finitely generated.

Sketch of proof.

Dymara-Januszkiewicz (2002): If $q^2 > \frac{1}{25}1764^n$, then $Isom(\Delta)$ has Kazhdan's property (T). Kazhdan's theorem plus lattice property implies that K also has property (T). But groups with property (T) are compactly generated, and K is discrete, hence finitely generated. □

→ Deep, non-elementary methods and a rather coarse bound.

Unitary forms are finitely generated

Theorem (Gramlich, H., Mühlherr, 2008)

Let G be a 2-spherical Kac-Moody group over a finite field $\mathbb{F}_q$, $q \geq 5$, and no fundamental rank 2 subgroup is isomorphic to $G_2(\mathbb{F}_q)$. Suppose θ is an involutory automorphism which interchanges the two conjugacy classes of Borel subgroups. If q is odd or θ semi-linear, then $\mathrm{Fix}_G(\theta)$ is finitely generated.

- Constant bound on q, does not depend on the rank n
- Restriction on G_2 residues: work in progress (H., Van Maldeghem)
- Works for almost arbitrary involutory automorphisms, with a price: q must be odd (or θ must be restricted again)
- Applies to other groups with root group datum, too

Unitary forms are finitely generated: Sketch of proof

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1. Define a suitable subcomplex $\mathcal{C}^\theta$ of the building (flip-flop system) such that $K.\mathcal{C}^\theta \subseteq \mathcal{C}^\theta$.
2. Choose a system X of representatives of the K-orbits on the maximal simplices in $\mathcal{C}^\theta$.
3. Show: $\mathcal{C}^\theta$ is pure and path connected. For this each possible rank 2 case is studied separately (H. and Van Maldeghem). Then apply a local to global argument.
4. For this reason, $G = \langle Stab_K(\sigma) \mid \sigma \text{ is non-empty face of } \sigma_0 \in X \rangle$.
5. Show: X is finite (follows from finiteness of maximal tori).
6. Show: Stabilizers in K of maximal simplices are finite.

Some more lattices

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As a nice side effect of all this and some other results from my thesis, the lattice result by Gramlich-Mühlherr can be adapted in a similar fashion:

Theorem

Let G be a 2-spherical Kac-Moody group over a finite field $\mathbb{F}_q$, with q sufficiently large and no fundamental rank 2 subgroup is isomorphic to $G_2(\mathbb{F}_q)$. Suppose θ is an involutory automorphism which interchanges the two conjugacy classes of Borel subgroups. If q is odd or θ semi-linear, then $\mathrm{Fix}_G(\theta)$ is a lattice in $\mathit{Isom}(\Delta)$.

References

Alice Devillers and Bernhard Mühlherr.
On the simple connectedness of certain subsets of buildings.
Forum Math., 19:955–970, 2007.

Aloysius G. Helminck and Shu Ping Wang.
On rationality properties of involutions of reductive groups.
Adv. Math., 99:26–96, 1993.

Max Horn.
Involutions of Kac-Moody groups.
PhD thesis, TU Darmstadt, 2008.

→ De Medts-Gramlich-H.: submitted; H.-Van Maldeghem plus Gramlich-H.-Mühlherr: in preparation; H.: Oberwolfach report

Ralf Gramlich and Andreas Mars.
Isomorphisms of unitary forms of Kac-Moody groups over finite fields
To appear in J. Algebra.