From Fischer spaces to (Lie) algebras

Max Horn

joint work with
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Technische Universität Braunschweig

Buildings 2010

Overview

1. 3-transposition groups and Fischer spaces
2. Algebras from Fischer spaces
3. Vanishing sets
4. Lie algebras
5. Some computations

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2 Algebras from Fischer spaces

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3-transposition groups

A class of 3-transpositions in a group G is a conjugacy class D of G such that

1. the elements of D are involutions and
2. for all $d, e \in D$ the order of de is equal to 1, 2 or 3.

G is called 3-transposition group if $G = \langle D \rangle$.

Examples

- Transpositions in $G = \mathrm{Sym}(n)$; $D = (12)^G$
- Transvections in $G = \mathrm{U}(n, 2)$; $D = d^G$ where

$$d = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$ (in GAP's version of this group)

- Fi_{22}, Fi_{23}, Fi_{24} (note: the simple group is Fi_{24}')

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Classification of 3-transpositions groups

- Fischer (around 1970) classified finite 3-transposition groups with no non-trivial normal solvable subgroups. $\rightsquigarrow$ classification of finite simple groups
- Cuypers and Hall (90s) classified all (possibly infinite) 3-transposition groups with trivial center, using geometric methods (Fischer spaces).
- Cuypers and Hall: If center is non-trivial, then $G/Z(G)$ is 3-transposition group with trivial center.

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Fischer spaces

- Throughout the rest of this talk, let D be a class of 3-transpositions generating a 3-transposition group G, and $Z(G) = 1$.
- $o(de) = 3 \iff de \neq ed \iff d \neq d^e = e^d \neq e$
- The Fischer space $\Pi(D)$ is the partial linear space with D as point set, and the triples $\{d, e, d^e\}$ as lines (when $o(de) = 3$).

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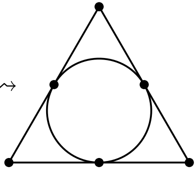
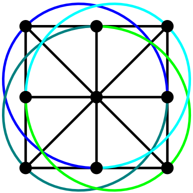
Characterizing Fischer spaces

Proposition (Buekenhout)

A partial linear space is a Fischer space if and only if every pair of intersecting lines generates a subspace isomorphic to the dual of an affine plane of order 2, or an affine plane of order 3.

$(\mathbb{F}_2^2)^{dual} \rightsquigarrow$ 

$\mathbb{F}_3^2 \rightsquigarrow$ 

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Algebras from Fischer spaces

- Denote by $\mathbb{F}_2 D$ the $\mathbb{F}_2$ vector space with basis D.
- Vectors are finite subsets of D; sum of two sets is their symmetric difference.
- Define the 3-transposition algebra $\mathcal{A}(D)$ with underlying vector space $\mathbb{F}_2 D$; multiplication is linear expansion of multiplication defined on $d, e \in D$ by

$$d * e := \begin{cases} d + e + e^d = \{d, e, e^d\} & \text{if } o(de) = 3 \\ 0 & \text{otherwise.} \end{cases}$$

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Group action on $\mathcal{A}(D)$

- G acts on $\mathcal{A}(D)$ by conjugation: $(d_1 + \ldots + d_n)^g = d_1^g + \ldots + d_n^g$.
- Let V be a subset of $\mathcal{A}(D)$. Then $I(V)$ denotes the ideal of $\mathcal{A}(D)$ generated by V.
- Goal: Compute Lie algebra quotients with a G-action.
- $G = \langle D \rangle$, so $I(V)$ is G-invariant if and only if for all $d \in D$, $X \in I(V)$ we have $X^d \in I(V)$.

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Vanishing sets

- For $d \in D$ set $A_d := d^\perp \setminus \{d\} = \{e \in D \mid de \neq ed\}$.

Lemma

Let X be a finite subset of D and $d \in D$. Then

$$d * X = X + X^d + (|A_d \cap X| \bmod 2)d.$$

- X is called vanishing set if $|A_d \cap X|$ is even for all $d \in D$.
- Examples: Empty set; point sets of finite maximal linear subspaces of $\Pi(D)$ (they have odd size); ...
- Vanishing ideals are ideals generated by vanishing sets.
- Any vanishing ideal I is G-invariant: If $X \in I$ then $d * X \in I$ hence $\{X^d \mid d \in D\} \in I$.

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The maximal vanishing ideal

Let $\mathcal{V}$ be the ideal of $\mathcal{A}$ generated by *all* vanishing subsets of D.

Lemma

1. *$\mathcal{V}$ equals the linear span of all vanishing subsets of D.*
2. *$\mathcal{V}$ is a proper ideal.*

Idea of proof:

1. Follows from the fact that $\mathcal{V}$ is G-invariant.
2. Define a symplectic form $\langle\cdot|\cdot\rangle$ on $\mathcal{A}(D)$: Set $\langle d|e\rangle = 1$ if $de \neq ed$ and 0 otherwise; extend linearly. This form is non-zero if there are lines (i.e. if G is non-abelian).

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Proper G-invariant ideals are vanishing

Lemma

Any G-invariant proper ideal of $\mathcal{A}(D)$ is contained in $\mathcal{V}$.

Proof:

Assume a G-invariant ideal I containing a non-vanishing set X. There is $d \in D$ such that $d * X = d + X + X^d \in I$. But I is G-invariant, thus $X^d \in I$ and so $d \in I$ and $d^G = D \subseteq I = \mathcal{A}$.

Proposition

Suppose Q is a simple quotient algebra of $\mathcal{A}(D)$. If G induces a group of automorphisms on Q, then Q is isomorphic to $\mathcal{A}(D)/\mathcal{V}$.

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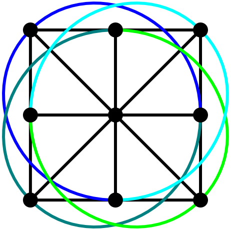

Lemma

Let I be an ideal of $\mathcal{A}(D)$. Then $\mathcal{A}(D)/I$ is a Lie algebra, if and only if every affine plane π of $\Pi(D)$ is in I.

If there are no affine planes, then $\mathcal{A}(D)$ is a Lie algebra and $\mathcal{A}(D)/\mathcal{V}$ is an abelian Lie algebra.

If affine planes are not vanishing sets, then no non-trivial quotient of $\mathcal{A}(D)$ is a Lie algebra.

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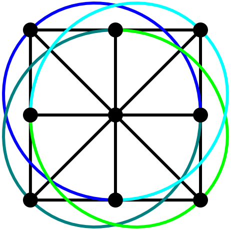

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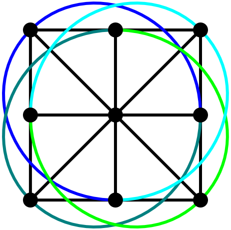

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Simple Lie algebras from 3-transposition groups

Theorem

Let D be a class of 3-transpositions generating a finite group G satisfying a certain irreducibility condition. Suppose $\mathcal{A}(D)/\mathcal{V}$ is a simple Lie algebra over $\mathbb{F}_2$ of dimension at least 2.

Then $\mathcal{A}(D)/\mathcal{V}$ is isomorphic to one of the following:

1. ${}^2A_n(2)$ *if* $G = 3^n : W(A_n)$ *or* $\mathrm{SU}_{n+1}(2)$*; for* $n = 5$ *also* $\mathrm{P}\Omega_6^-(3)$.
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3. $D_n(2)$ *if* $G = 3^n : W(D_n)$ *and* n *even; for* $n = 4$ *also* $\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}_3$.
4. ${}^2E_6(2)$ *if* $G = 3^6 : W(E_6)$ *or* $\mathrm{P}\Omega_7(3)$ *or* Fi_{22}.
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4. ${}^2E_6(2)$ *if* $G = 3^6 : W(E_6)$ *or* $\mathrm{P}\Omega_7(3)$ *or* Fi_{22}.
5. $E_7(2)$, $E_8(2)$ *if* $G = 3^n : W(E_n)$.

Simple Lie algebras from 3-transposition groups

Theorem

Let D be a class of 3-transpositions generating a finite group G satisfying a certain irreducibility condition. Suppose $\mathcal{A}(D)/\mathcal{V}$ is a simple Lie algebra over $\mathbb{F}_2$ of dimension at least 2.

Then $\mathcal{A}(D)/\mathcal{V}$ is isomorphic to one of the following:

1. ${}^2A_n(2)$ *if* $G = 3^n : W(A_n)$ *or* $\mathrm{SU}_{n+1}(2)$; *for* $n = 5$ *also* $\mathrm{P}\Omega_6^-(3)$.
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3. $D_n(2)$ *if* $G = 3^n : W(D_n)$ *and n even; for* $n = 4$ *also* $\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}_3$.
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Overview

1. 3-transposition groups and Fischer spaces
2. Algebras from Fischer spaces
3. Vanishing sets
4. Lie algebras
5. Some computations

Unitary groups

L_{max} denotes the maximal Lie algebra quotient of $\mathcal{A}(D)$.

G	$\lvert D\rvert$	$\dim L_{max}$	$\dim \mathcal{A}(D)/\mathcal{V}$
$U_2(2)$	3	3	2
$U_3(2)$	9	8	8
$U_4(2)$	45	30	14
$U_5(2)$	165	45	24
$U_6(2)$	693	78	34
$U_7(2)$	2709	119	48
$U_8(2)$	10789	176	62
$U_9(2)$	43356	249	80
$U_{10}(2)$	174933	340	98
$U_{11}(2)$	?	?	120
$U_n(2)$	$\frac{1}{6}(4^n + (-2)^n - 2)$	???	$n^2 - 2 + (n \bmod 2)$

In fact, $\mathcal{A}(D)/\mathcal{V} \cong {}^2A_n(2)$ holds.

Sporadic cases

L_{max} denotes the maximal Lie algebra quotient of $\mathcal{A}(D)$.

G	$\|D\|$	$\dim L_{max}$	$\dim \mathcal{A}(D)/\mathcal{V}$
$O^+(8,2) : \mathrm{Sym}_3$	360	52	26
$O^+(8,3) : \mathrm{Sym}_3$	3240	0	782
Fi_{22}	3510	78	78
Fi_{23}	31671	0	782
Fi_{24}	306936	0	3774

For $O^+(8,2) : \mathrm{Sym}_3$ we get the simple Lie algebra $D_4(2)$ and for Fi_{22} the simple Lie algebra ${}^2E_6(2)$.

In the other cases, we do not get Lie algebras, but still a non-trivial algebra structure.

Thank you!