On the Phan system of the Schur cover of $\mathrm{SU}(4, 3^2)$

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Abstract

This article is part of the program described in [3]. We study the Phan amalgams and their universal completions that occur for $q = 3$ in rank $n = 3$ for the diagram $A_3 = D_3$, corresponding to $\mathrm{SU}(4, 3^2) \cong \mathrm{Spin}^-(6, 3)$. We show that the associated geometries admit universal 9-fold coverings, by showing that the universal completion of the Phan amalgam is the central extension of $\mathrm{SU}(4, 3^2)$ by its Schur multiplier. This information provides the last missing piece of information in the full classification of Phan amalgams and their universal completions for A_n and D_n.

1 Introduction

The purpose of this paper is to shed some light on the last open problem in the classification of Phan amalgams for A_n and D_n, namely the case $q = 3$, and there in particular the rank 3 situation. In the theory of Phan amalgams, higher rank cases are reduced to lower rank cases by induction, which makes it crucial to understand the low rank cases first. For an overview of the general program, refer to [3] and also more recently [13]. The case A_n is dealt with in [5], the case D_n in [12]. The program Phan Theory is named after Kok-Wee Phan who in [15], [16] gave precursors of Theorem 1.2 below and similiar theorems dealing with D_n and E_n. Phan's results entered the classification of the finite simple groups via Aschbacher's paper [1].

We will focus on the A_3 viewpoint (that is, we will consider the geometry constructed for A_n, and the resulting amalgam, not that for D_n).

For the convenience of the reader, we reproduce the Phan theorem for A_n here. Formulated in terms of the geometry $\mathcal{G}_A^{\mathrm{herm}}(n, q)$ (defined in the next section) we have the following (taken from [9], originally from [5]):

Theorem 1.1. *Suppose $n \geq 3$, q a prime power. Then the geometry $\mathcal{G}_A^{\mathrm{herm}}(n, q)$ is simply connected unless $(n, q) = (3, 2)$ or $(n, q) = (3, 3)$.*

This has important group theoretic consequences (one of the reasons one is interested in Phan theorems is their use in the revised classification of finite simple groups), from *loc.cit.*:

Theorem 1.2. *Let q be a prime power, consider the flag-transitive action of $\mathrm{SU}(n + 1, q^2)$ on $\mathcal{G}_A^{\mathrm{herm}}(n, q)$. Then the following hold.*

(i) *If $n \geq 3$, $q \geq 4$, then $\mathrm{SU}(n + 1, q^2)$ is the universal completion of the amalgam $\mathcal{A}_{(2)}$ of rank two parabolics.*

(ii) *If $n \geq 4$, then $\mathrm{SU}(n + 1, q^2)$ is the universal completion of the amalgam $\mathcal{A}_{(3)}$ of rank three parabolics.*

Neither theorem makes statements about the case $(n, q) = (3, 2)$. This is because for $(n, q) = (2, 2)$, the geometry $\mathcal{G}_A^{\mathrm{herm}}(2, 2)$ is not connected. As a consequence, the geometries $\mathcal{G}_A^{\mathrm{herm}}(n, 2)$ are not residually connected, and the reduction argument used in the proof fails.

Likewise, neither theorem makes statements about the case $(n, q) = (3, 3)$. Despite the geometry $\mathcal{G}_A^{\mathrm{herm}}(3, 3)$ being (residually) connected, it was still known that the theorems do not hold for it either, by a simple argument due to Richard Lyons (c.f. [9, section 4.4]). However, the precise structure of the universal cover of the geometry, resp. the universal completion of the Phan amalgam, was not known. In this paper, we present some insights on the universal cover of the geometry $\mathcal{G}_A^{\mathrm{herm}}(3,3)$ and the universal completion of the corresponding amalgam. All together we prove the following result:

Main Theorem. *The universal completion of the standard Phan amalgam of* $\mathrm{SU}(4,3^2)$ *is isomorphic to* $3^2.\mathrm{SU}(4,3^2)$*, its Schur cover. The geometry* $\mathcal{G}_A^{\mathrm{herm}}(3,3)$ *admits a 9-fold universal cover. In particular, it is not simply connected.*

Lastly, Theorem 1.2 distinguishes between rank 3 and rank 4 for $q \in \{2,3\}$, precisely because of the two special cases we just mentioned; they effectively mean that in the proof of the theorem, one can only reduce down to rank three, but not rank two. It is an interesting question to ask which group the amalgam of rank two parabolics describes for $n \geq 4$ and $q = 3$. The answer for this is unknown to the author, but empirical results seem to indicate a high degree covering (several thousand fold), or maybe even an infinite one.

In Section 2, we will briefly describe the geometrical setting, mostly referring to previous publications for details. In Section 3, we analyze the amalgam of rank two parabolics for $(n,q) = (3,3)$ by applying methods from computer algebra, similar to those previously used by the author in [10], [14] (for C_n) and [11] (for B_n). It is assumed throughout the paper that the reader is familiar with the concept of amalgams (refer to [17] for an introduction to the subject.)

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2 Geometrical setting

In this section we describe the geometry dealt with in the Theorem. For an introduction to synthetic geometry, refer to [2].

Fix a dimension n and an odd prime power q (note that in this article we are primarily concerned with the case $n = q = 3$).

Let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_{q^2}$, equipped with a nondegenerate hermitian form $(\cdot,\cdot)$. We define the pregeometry $\mathcal{G}_A^{\mathrm{herm}}(n,q)$ over the set $\{1,\dots,n\}$ whose elements of type k are the nondegenerate subspaces of V of dimension k. Incidence in $\mathcal{G}_A^{\mathrm{herm}}(n,q)$ is defined by symmetrized containment. By Lemma 1.2 in [5], $\mathcal{G}_A^{\mathrm{herm}}(n,q)$ is actually a geometry.

We now consider a hermitian basis $e_1,\dots,e_{n+1}$ of V, i.e. $(\cdot,\cdot)$ becomes the usual sesquilinear scalar product w.r.t. this basis. Then the nondegenerate subspaces $V_1 := \langle e_1\rangle, V_2 := \langle e_1, e_2\rangle, \dots, V_n := \langle e_1,\dots,e_n\rangle$ form a maximal flag F of $\mathcal{G}_A^{\mathrm{herm}}(n,q)$. By Section 4 in [5], the maximal parabolics M_i with respect to this flag (i.e. the stabilizers of the V_i) are isomorphic to

$$\mathrm{S}(\mathrm{GU}_i(q^2) \times \mathrm{GU}_{n+1-i}(q^2)).$$

In particular, M_i contains a normal subgroup M_i^0 of the form

$$\mathrm{SU}_i(q^2) \times \mathrm{SU}_{n+1-i}(q^2)$$

with $M_i = M_i^0 T$. This is a maximal so-called *stripped parabolic* (again see [5]).

The connection between the simple connectedness of this geometry and the amalgams of parabolics of the group $\mathrm{SU}(n+1,q^2)$ is established by Tits' famous Lemma (see [18]).

Lemma 2.1 (Tits' Lemma)**.** *Suppose a group G acts flag-transitively on a geometry $\mathcal{G}$, and let $\mathcal{A}$ be the amalgam of parabolics associated with some maximal flag F of $\mathcal{G}$. Then G is the universal completion of the amalgam $\mathcal{A}$ if and only if $\mathcal{G}$ is simply connected.*

3 Computations

In order to analyze the case $n = q = 3$ for A_n resp. D_n, we used the computer algebra system GAP [7], carefully turning the above theoretical description into computer code. The key idea here is to choose a particularly nice set of generators of $\mathrm{SU}(4, 3^2)$, namely one which can be used to describe the parabolics w.r.t. F and their intersections in a nice way (details will be given below). This makes it very easy to obtain a finite presentation of the universal completion of the amalgam we are interested in.

We start by giving a representation of $\mathrm{SU}(4, 3^2)$ from which it is easy to extract the maximal (stripped) parabolics. Let z denote a primitive element in $\mathbb{F}_9$ over $\mathbb{F}_3$ with minimal polynomial $x^2 - x - 1$. Note that $(z^2)^3 = z^6 = (z^2)^{-1}$. We define the following matrices

$$U : \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & z^5 & z^7 \\ & & z & z^7 \end{pmatrix}, \qquad V : \begin{pmatrix} 1 & & & \\ & z^5 & z^7 & \\ & z & z^7 & \\ & & & 1 \end{pmatrix}, \qquad W : \begin{pmatrix} z^5 & z^7 & & \\ z & z^7 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

In addition we use the diagonal matrices $D_1 := \mathrm{diag}(1, 1, z^2, z^6)$, $D_2 := \mathrm{diag}(1, z^2, z^6, 1)$, $D_3 := \mathrm{diag}(z^2, z^6, 1, 1)$, which generate the stabilizer of the maximal flag F.

Lemma 3.1. *The matrices U, V, W and $D_i, 1 \le i \le 3$ generate $\mathrm{SU}(4, 3^2)$.*

Proof. Using GAP or another computer algebra system one easily verifies that the matrices are all invertible, generate a group of the correct size and preserve $(\cdot, \cdot)$. □

Next, we see that the stripped parabolics can be generated by these generators:

Lemma 3.2. *Each maximal stripped parabolic in $\mathrm{SU}(4, 3^2)$ w.r.t. the flag F is generated by the matrices, and has the size and index, as specified in the following table.*

stabilizer	*stabilized flag*	*isomorphism type*	*generators*	*size*	*index*
M^0_{123}	F	C_4^3	D_1, D_2, D_3	64	204120
M^0_1	$\langle e_1 \rangle$	$\mathrm{SU}(3, 3^2)$	D_1, D_2, U, V	6048	2160
M^0_2	$\langle e_1, e_2 \rangle$	$\mathrm{SU}(2, 3^2) \times \mathrm{SU}(3, 3^2)$	D_1, D_3, U, W	576	22680
M^0_3	$\langle e_1, e_2, e_3 \rangle$	$\mathrm{SU}(3, 3^2)$	D_2, D_3, V, W	6048	2160

Furthermore, the pairwise intersections of the stabilizers are generated by the intersections of the generating sets given above.

Proof. The isomorphism types, and thus the sizes and indices in the table follow from the characterization of the stripped parabolics in the previous sections. The given generators obviously stabilize the associated flags, thus generate subgroups of the M^0_i. Using GAP or another computer algebra system, we verify that the groups have the correct sizes. Finally, the claim about the intersections of the parabolics (i.e. the double stabilizers) follows similarly. □

Based on the above table, we determine (again with the help of GAP) finite presentations of the maximal parabolics on generators d_1, d_2, d_3, u, v, w (corresponding to the matrix generators D_1, D_2, D_3, U, V, W). From this we get a finite presentation of the universal completion H of the amalgam of the maximal parabolics, by virtue of the generator intersection property, as follows: Take generators d_1, d_2, d_3, u, v, w, and as relators the union of the relators of the three maximal parabolics. In our case, GAP yielded the following presentation:

$$d_1^4, d_2^4, d_3^4, u^3, v^3, w^3, [d_1, d_2], [d_1, d_3], [d_2, d_3], [d_1, w], [d_3, u], [u, w], [d_2^2, v], [d_3^2, w],$$
$$d_1^{-1}d_2ud_2^{-1}u,\ d_2^{-1}vd_1^{-1}vd_1,\ d_2^{-1}vd_3^{-1}vd_3,\ d_3^{-1}d_2wd_2^{-1}w,\ (ud_1)^3d_1^2,\ (wd_3)^3d_3^2,$$
$$uvu^{-1}vuv^{-1},\ vwvw(wvwv)^{-1},\ vwd_2^{-1}vw^{-1}v^{-1}wd_3^{-1},\ d_2^{-1}v^{-1}ud_1uvd_2uvd_2^{-1}u^{-1},$$
$$(uvud_2^2)^3,\ (d_2vwd_2wd_3^{-1}v^{-1})^3,\ (d_3vwd_2^{-1}v^{-1})^3,\ d_2uvd_2uv^{-1}u^{-1}d_2^{-1}v^{-1}u^{-1}d_2^{-1}v,$$
$$d_1^2vud_2d_1vd_2d_1^2u^{-1}vd_2u^{-1},\ vwd_2wd_3^{-1}v^{-1}w^{-1}vd_3w^{-1}d_2^{-1}w^{-1}v^{-1}w^{-1}d_3,$$
$$w^{-1}v^{-1}d_2^{-1}w^{-1}d_3vwd_2^{-1}v^{-1}d_2^{-1}vw^{-1}d_3^{-1}vwd_3^{-1},\ d_2^{-1}v^{-1}d_1^2uv^{-1}u^{-1}d_2^2d_1^2uvd_1u^{-1}d_2^{-1}u^{-1}v^{-1}u$$

Note that $G := \mathrm{SU}(4, 3^2)$ is a quotient of H. We now describe the exact structure of H.

Lemma 3.3. *Let $\phi : H \to G$ be the canonical group epimorphism which maps the generators d_1, d_2, d_3, u, v, w of H onto the generators D_1, D_2, D_3, U, V, W of G. Then $K := \ker\phi \cong C_3^2$ and K is central in H.*

Proof. Regard the subgroup $M_1' := \langle d_1, d_2, u, v\rangle$ of H. By the construction of H, $\phi|_{M_1'}$ is an isomorphism between M_1' and M_1^0. Using GAP and ACE (the Advanced Coset Enumerator [8]), we compute a coset table w.r.t. M_1', which yields that $|H : M_1'| = 19440$, and thus $|K| = |H : G| = |H : M_1'|/|G : M_1^0| = 9$. From the coset table we also obtain a faithful permutation presentation of H, with which it becomes a trivial matter to establish the remaining claims with computer help. □

Proposition 3.4. *The group H is isomorphic to a non-split central extension of* $\mathrm{SU}(4, 3^2)$ *by K, i.e. the following sequence is exact and non-split:*

$$1 \to K \to H \xrightarrow{\phi} \mathrm{SU}(4, 3^2) \to 1.$$

Proof. The only thing left to show is that the extension is non-split. This is equivalent to showing that there are no complements to K in H. But we already have nice permutation presentations of H and K. Hence, confirming this amounts to invoking the GAP command `Complementclasses(H,K)`, which confirms that indeed no complements exist. □

Since $\mathrm{SU}(4, 3^2)$ is perfect, the Main Theorem is proved.

It would be interesting to further study the resulting geometry. A possible starting point for that might be [6], adapted to work in characteristic three.

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